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EQUATIONS OF DEFORMATION OF AN ELASTIC INHOMOGENEOUS LAMINATED BODY OF REVOLUTION

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Equations governing deformation of an elastic inhomogeneous laminated body of revolution are proposed. Each layer is a domain bounded by convex equidistant surfaces of revolution.

Key words: *laminated, elastic, body of revolution, Legendre polynomials, equidistant.*

Introduction. Various methods for constructing the theory of elastic deformation of multilayered structures are considered in [1–4].

In the present paper, equations governing elastic deformation of a laminated body of revolution are constructed with the use of the results of [5–7] obtained by employing several approximations of each unknown function in the form of truncated series in Legendre polynomials. This approach allows one to adequately formulate the conjugation conditions for stresses and displacements at the interlayer surfaces. Some problems of elastic deformation of laminated structures [7–9] are solved by the method proposed.

1. Curvilinear Coordinates. Let S be a sufficiently smooth closed convex surface and the origin O of the coordinate system (x, y, z) lie inside S at the z axis (Fig. 1). The surface S is formed by revolution of a convex curve L located in the zr plane, where $r = (x^2 + y^2)^{1/2}$. The curve L intersects the z axis at the right angle, and the curvature radius at each point of the curve L is equal to or greater than ρ_* (Fig. 2).

We write the equation of the curve L in the form

$$r = \hat{r}(\gamma) = \frac{dF(\gamma)}{d\gamma}\cos\gamma + F(\gamma)\sin\gamma, \qquad z = \hat{z}(\gamma) = \frac{dF(\gamma)}{d\gamma}\sin\gamma - F(\gamma)\cos\gamma.$$

Here γ is the angle between the tangent line and the r axis, $F(\gamma)$ is the support function of the contour L (distance between the point O and the tangent line). It is obvious that $\hat{r}(\gamma) \geqslant 0$ for $0 \leqslant \gamma \leqslant \pi$. The curvature radius of the curve L is

$$\rho = \rho(\gamma) = F(\gamma) + \frac{d^2F(\gamma)}{d\gamma^2} \geqslant \rho_* > 0.$$

We write the equations of the surface S as

$$x = x_S(\beta,\gamma) = \Big(\frac{dF(\gamma)}{d\gamma}\cos\gamma + F(\gamma)\sin\gamma\Big)\cos\beta,$$

$$y = y_S(\beta,\gamma) = \Big(\frac{dF(\gamma)}{d\gamma}\cos\gamma + F(\gamma)\sin\gamma\Big)\sin\beta, \qquad z = z_S(\beta,\gamma) = \frac{dF(\gamma)}{d\gamma}\sin\gamma - F(\gamma)\cos\gamma.$$

We consider the orthogonal curvilinear coordinate system (α, β, γ):

$$x = x(\alpha,\beta,\gamma) = \Big(\frac{dF(\gamma)}{d\gamma}\cos\gamma + (F(\gamma)+\alpha)\sin\gamma\Big)\cos\beta, \quad y = y(\alpha,\beta,\gamma) = \Big(\frac{dF(\gamma)}{d\gamma}\cos\gamma + (F(\gamma)+\alpha)\sin\gamma\Big)\sin\beta,$$

$$z = z(\alpha,\beta,\gamma) = \frac{dF(\gamma)}{d\gamma}\sin\gamma - (F(\gamma)+\alpha)\cos\gamma \qquad (\alpha \geqslant 0, \qquad 0 \leqslant \gamma \leqslant \pi, \qquad 0 \leqslant \beta < 2\pi). \tag{1.1}$$

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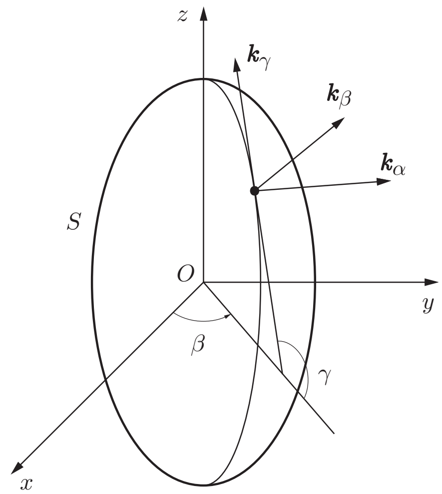

Fig. 1

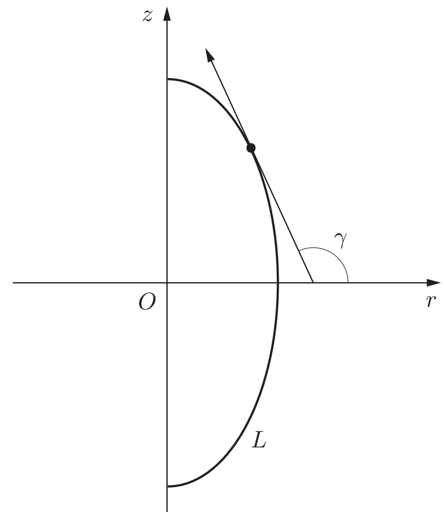

Fig. 2

The Jacobian of the coordinate transformation $J(\alpha, \beta, \gamma)$ does not change its sign:

$$J(\alpha, \beta, \gamma) = D(x, y, z)/D(\alpha, \beta, \gamma) = (\rho(\gamma) + \alpha)(\hat{r}(\gamma) + \alpha \sin \gamma) \geqslant 0.$$

It follows from (1.1) that the surface $\alpha = \text{const}$ is a surface equidistant to the surface S. The unit vectors of the coordinate lines (see Fig. 1) have the form

$$\boldsymbol{k}_\alpha = (\sin\gamma \cos\beta, \sin\gamma \sin\beta, -\cos\gamma), \qquad \boldsymbol{k}_\beta = (-\sin\beta, \cos\beta, 0),$$

$$\boldsymbol{k}_\gamma = (\cos\gamma \cos\beta, \cos\gamma \sin\beta, -\sin\gamma).$$

2. Equations of the Linear Theory of Elasticity in the Curvilinear System of Coordinates $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. We formulate the problem of the linear theory of elasticity in an orthogonal coordinate system (α, β, γ). The stresses $\sigma_{\alpha\alpha}$, $\sigma_{\alpha\beta}$, $\sigma_{\alpha\gamma}$, $\sigma_{\beta\beta}$, $\sigma_{\beta\gamma}$, and $\sigma_{\gamma\gamma}$ satisfy the equations of equilibrium

$$\frac{\partial \hat{\boldsymbol{t}}_\alpha}{\partial \alpha} + \frac{\partial \hat{\boldsymbol{t}}_\beta}{\partial \beta} + \frac{\partial \hat{\boldsymbol{t}}_\gamma}{\partial \gamma} = 0.$$

Here, we have

$$\hat{\boldsymbol{t}}_\alpha = H_\gamma H_\beta (\sigma_{\alpha\alpha} \boldsymbol{k}_\alpha + \sigma_{\alpha\beta} \boldsymbol{k}_\beta + \sigma_{\alpha\gamma} \boldsymbol{k}_\gamma),$$

$$\hat{\boldsymbol{t}}_\beta = H_\alpha H_\gamma (\sigma_{\alpha\beta} \boldsymbol{k}_\alpha + \sigma_{\beta\beta} \boldsymbol{k}_\beta + \sigma_{\beta\gamma} \boldsymbol{k}_\gamma), \qquad \hat{\boldsymbol{t}}_\gamma = H_\alpha H_\beta (\sigma_{\alpha\gamma} \boldsymbol{k}_\alpha + \sigma_{\beta\gamma} \boldsymbol{k}_\beta + \sigma_{\gamma\gamma} \boldsymbol{k}_\gamma),$$

$$H_\alpha = 1, \qquad H_\beta = \hat{r}(\gamma) + \alpha - \sin\gamma, \qquad H_\gamma = \rho(\gamma) + \alpha.$$

The strain tensor is determined in terms of the displacement vector $\boldsymbol{U}$:

$$e_{\alpha\alpha} = \frac{\boldsymbol{k}_\alpha}{H_\alpha} \cdot \frac{\partial \boldsymbol{U}}{\partial \alpha}, \quad e_{\beta\beta} = \frac{\boldsymbol{k}_\beta}{H_\beta} \cdot \frac{\partial \boldsymbol{U}}{\partial \beta}, \quad e_{\gamma\gamma} = \frac{\boldsymbol{k}_\gamma}{H_\gamma} \cdot \frac{\partial \boldsymbol{U}}{\partial \gamma}, \qquad 2e_{\alpha\beta} = \frac{\boldsymbol{k}_\alpha}{H_\beta} \cdot \frac{\partial \boldsymbol{U}}{\partial \beta} + \frac{\boldsymbol{k}_\beta}{H_\alpha} \cdot \frac{\partial \boldsymbol{U}}{\partial \alpha},$$

$$2e_{\alpha\gamma} = \frac{\boldsymbol{k}_\alpha}{H_\gamma} \cdot \frac{\partial \boldsymbol{U}}{\partial \gamma} + \frac{\boldsymbol{k}_\gamma}{H_\alpha} \cdot \frac{\partial \boldsymbol{U}}{\partial \alpha}, \qquad 2e_{\beta\gamma} = \frac{\boldsymbol{k}_\beta}{H_\gamma} \cdot \frac{\partial \boldsymbol{U}}{\partial \gamma} + \frac{\boldsymbol{k}_\gamma}{H_\beta} \cdot \frac{\partial \boldsymbol{U}}{\partial \beta}. \tag{2.1}$$

The stresses and strains are related by Hooke's law

$$\sigma_{\alpha\alpha} = 2\mu e_{\alpha\alpha} + \lambda e, \quad \sigma_{\beta\beta} = 2\mu e_{\beta\beta} + \lambda e, \quad \sigma_{\gamma\gamma} = 2\mu e_{\gamma\gamma} + \lambda e,$$

$$\sigma_{\alpha\beta} = 2\mu e_{\alpha\beta}, \quad \sigma_{\alpha\gamma} = 2\mu e_{\alpha\gamma}, \quad \sigma_{\beta\gamma} = 2\mu e_{\beta\gamma}, \qquad e = e_{\alpha\alpha} + e_{\beta\beta} + e_{\gamma\gamma}, \tag{2.2}$$

where λ and μ are the elastic moduli.

3. Equations of Elastic Deformation of an Oval Shell of Revolution. We consider a shell of revolution of thickness $2h$ that occupies the volume V and is bounded by the coordinate surfaces α_1 and α_2 such that $0 < \alpha_1 < \alpha_2 = \alpha_1 + 2h$. We introduce the coordinate $\xi \in [-1, 1]$ in the α direction so that $\alpha = \alpha_0 + h\xi$ and $\alpha_0 = (\alpha_1 + \alpha_2)/2$.

The unknown functions $\boldsymbol{U}$, $\hat{\boldsymbol{t}}_\alpha$, $\hat{\boldsymbol{t}}_\beta$, and $\hat{\boldsymbol{t}}_\gamma$ can be written in the form of series in Legendre polynomials:

$$\boldsymbol{U} = \sum_{k=0}^{\infty} [\boldsymbol{U}]^k p_k(\xi), \qquad \hat{\boldsymbol{t}}_i = \sum_{k=0}^{\infty} [\hat{\boldsymbol{t}}_i]^k p_k(\xi). \tag{3.1}$$

Here $p_k(\xi)$ are the orthogonal Legendre polynomials; $[\boldsymbol{U}]^k$ and $[\hat{\boldsymbol{t}}_i]^k$ are expansion coefficients depending on the coordinates β and γ:

$$[\boldsymbol{U}]^k = \frac{1+2k}{2} \int_{-1}^{1} \boldsymbol{U} p_k \, d\xi, \qquad [\hat{\boldsymbol{t}}_i]^k = \frac{1+2k}{2} \int_{-1}^{1} \hat{\boldsymbol{t}}_i p_k \, d\xi.$$

The surface $\xi = 0$ is the mid-surface of the shell. In the orthogonal coordinate system (α, β, γ), the following relations are valid:

$$H_\beta = A_\beta (1 + \xi h / R_\beta), \qquad H_\gamma = A_\gamma (1 + \xi h / R_\gamma),$$

$$R_\beta = \frac{A_\beta}{\sin\gamma} = \frac{dF}{d\gamma} \cot\gamma + F(\gamma) + \alpha_0, \quad R_\gamma = A_\gamma = \frac{d^2 F}{d\gamma^2} + F(\gamma) + \alpha_0,$$

where A_β and A_γ are the Lamé coefficients of the mid-surface; R_β and R_γ are the principal curvature radii of the mid-surface.

In accordance with [5–7], we approximate the stresses by the truncated series (3.1):

$$\hat{\boldsymbol{t}}_\beta \simeq A_\gamma (\boldsymbol{N}_\beta p_0/(2h) + 3\boldsymbol{M}_\beta p_1/(2h^2)), \quad \hat{\boldsymbol{t}}_\gamma \simeq A_\beta (\boldsymbol{N}_\gamma p_0/(2h) + 3\boldsymbol{M}_\gamma p_1/(2h^2)),$$

$$\hat{\boldsymbol{t}}_\alpha \simeq A_\beta A_\gamma [\boldsymbol{P}_0 p_0 + \Delta\boldsymbol{P} p_1 + (p_2 - p_0)(\boldsymbol{k}_\alpha \times (\boldsymbol{P}_0 \times \boldsymbol{k}_\alpha) - \boldsymbol{Q}/(2h))],$$

$$\Delta\boldsymbol{P} = (\boldsymbol{P}^+ - \boldsymbol{P}^-)/2, \qquad \boldsymbol{P}_0 = (\boldsymbol{P}^+ + \boldsymbol{P}^-)/2.$$

Here

$$\boldsymbol{N}_\beta = h \int_{-1}^{1} \Big(\sigma_{\alpha\beta}\boldsymbol{k}_\alpha + \sigma_{\beta\beta}\boldsymbol{k}_\beta + \Big(1 + \frac{h\xi}{R_\gamma}\Big)\sigma_{\beta\gamma}\boldsymbol{k}_\gamma\Big) d\xi, \qquad \boldsymbol{N}_\gamma = h \int_{-1}^{1} \Big(\sigma_{\alpha\gamma}\boldsymbol{k}_\alpha + \Big(1 + \frac{h\xi}{R_\beta}\Big)\sigma_{\beta\gamma}\boldsymbol{k}_\beta + \sigma_{\gamma\gamma}\boldsymbol{k}_\gamma\Big) d\xi,$$

$$\boldsymbol{M}_\beta = h^2 \int_{-1}^{1} \xi(\sigma_{\beta\beta}\boldsymbol{k}_\beta + \sigma_{\beta\gamma}\boldsymbol{k}_\gamma) \, d\xi, \qquad \boldsymbol{M}_\gamma = h^2 \int_{-1}^{1} \xi(\sigma_{\beta\gamma}\boldsymbol{k}_\beta + \sigma_{\gamma\gamma}\boldsymbol{k}_\gamma) \, d\xi, \tag{3.2}$$

$$\boldsymbol{Q} = h \int_{-1}^{1} (\sigma_{\alpha\beta}\boldsymbol{k}_\beta + \sigma_{\alpha\gamma}\boldsymbol{k}_\gamma) \, d\xi, \qquad \boldsymbol{P}^\pm = (\sigma_{\alpha\alpha}\boldsymbol{k}_\alpha + \sigma_{\alpha\beta}\boldsymbol{k}_\beta + \sigma_{\alpha\gamma}\boldsymbol{k}_\gamma)\Big|_{\xi=\pm1}.$$

The displacements $\boldsymbol{U} = U_\alpha\boldsymbol{k}_\alpha + U_\beta\boldsymbol{k}_\beta + U_\gamma\boldsymbol{k}_\gamma$ are approximated by the truncated series (3.1)

$$\begin{gathered} \boldsymbol{k}_\alpha \times (\boldsymbol{U} \times \boldsymbol{k}_\alpha) = \boldsymbol{v} p_0 + \boldsymbol{\psi} p_1 + (\boldsymbol{v}_0 - \boldsymbol{v}) p_2 + (\Delta\boldsymbol{v} - \boldsymbol{\psi}) p_3, \qquad \boldsymbol{U} \cdot \boldsymbol{k}_\alpha = W p_0 + \Delta W p_1 + (W_0 - W) p_2, \\ \Delta\boldsymbol{v} = (\boldsymbol{v}^+ - \boldsymbol{v}^-)/2, \qquad \boldsymbol{v}_0 = (\boldsymbol{v}^+ + \boldsymbol{v}^-)/2, \qquad \Delta W = (W^+ - W^-)/2, \qquad W_0 = (W^+ + W^-)/2. \end{gathered} \tag{3.3}$$

Here

$$\boldsymbol{v} = \frac{1}{2} \int_{-1}^{1} (\boldsymbol{k}_\beta U_\beta + \boldsymbol{k}_\gamma U_\gamma) \, d\xi, \qquad \boldsymbol{\psi} = \frac{1}{2} \int_{-1}^{1} (\boldsymbol{k}_\beta U_\beta + \boldsymbol{k}_\gamma U_\gamma) \xi \, d\xi,$$

$$W = \frac{1}{2} \int_{-1}^{1} \boldsymbol{k}_\alpha \cdot \boldsymbol{U} \, d\xi, \qquad W^\pm = \boldsymbol{k}_\alpha \cdot \boldsymbol{U}\Big|_{\xi=\pm1}, \qquad \boldsymbol{v}^\pm = (\boldsymbol{k}_\beta U_\beta + \boldsymbol{k}_\gamma U_\gamma)\Big|_{\xi=\pm1}.$$

The strains (2.1) are approximated by the truncated series

$$e_{\alpha\alpha} = \varepsilon_{\alpha\alpha}p_0 + \chi_{\alpha\alpha}p_1, \qquad e_{\beta\beta} = \varepsilon_{\beta\beta}p_0 + \chi_{\beta\beta}p_1, \qquad e_{\gamma\gamma} = \varepsilon_{\gamma\gamma}p_0 + \chi_{\gamma\gamma}p_1,$$
$$e_{\beta\gamma} = \varepsilon_{\beta\gamma}p_0 + \chi_{\beta\gamma}p_1, \qquad e_{\alpha\beta} = \varepsilon_{\alpha\beta}p_0 + \chi_{\alpha\beta}p_1 + \omega_{\alpha\beta}p_2, \qquad e_{\alpha\gamma} = \varepsilon_{\alpha\gamma}p_0 + \chi_{\alpha\gamma}p_1 + \omega_{\alpha\gamma}p_2. \tag{3.4}$$

Here

$$\varepsilon_{\alpha\alpha} = \Delta W/h, \qquad \chi_{\alpha\alpha} = 3(W_0 - W)/h, \qquad \varepsilon_{\beta\beta} = \frac{\boldsymbol{k}_\beta}{A_\beta}\cdot\frac{\partial \boldsymbol{v}}{\partial\beta} + \frac{1}{R_\beta}W, \qquad \chi_{\beta\beta} = \frac{\boldsymbol{k}_\beta}{A_\beta}\cdot\frac{\partial\boldsymbol{\psi}}{\partial\beta},$$
$$\varepsilon_{\gamma\gamma} = \frac{\boldsymbol{k}_\gamma}{A_\gamma}\cdot\frac{\partial \boldsymbol{v}}{\partial\gamma} + \frac{1}{R_\gamma}W, \qquad \chi_{\gamma\gamma} = \frac{\boldsymbol{k}_\gamma}{A_\gamma}\cdot\frac{\partial\boldsymbol{\psi}}{\partial\gamma}, \qquad 2\varepsilon_{\beta\gamma} = \frac{\boldsymbol{k}_\gamma}{A_\beta}\cdot\frac{\partial \boldsymbol{v}}{\partial\beta} + \frac{\boldsymbol{k}_\beta}{A_\gamma}\cdot\frac{\partial \boldsymbol{v}}{\partial\gamma}, \tag{3.5}$$
$$2\chi_{\beta\gamma} = \frac{1}{R_\gamma}\frac{\boldsymbol{k}_\gamma}{A_\beta}\cdot\frac{\partial \boldsymbol{v}}{\partial\beta} + \frac{1}{R_\beta}\frac{\boldsymbol{k}_\beta}{A_\gamma}\cdot\frac{\partial \boldsymbol{v}}{\partial\gamma} + \frac{\boldsymbol{k}_\gamma}{A_\beta}\cdot\frac{\partial\boldsymbol{\psi}}{\partial\beta} + \frac{\boldsymbol{k}_\beta}{A_\gamma}\cdot\frac{\partial\boldsymbol{\psi}}{\partial\gamma}, \qquad 2\varepsilon_{\alpha\beta} = \frac{\boldsymbol{k}_\alpha}{A_\beta}\cdot\frac{\partial \boldsymbol{v}}{\partial\beta} + \frac{1}{A_\beta}\frac{\partial W}{\partial\beta} + \frac{1}{h}\boldsymbol{k}_\beta\cdot\Delta\boldsymbol{v},$$
$$2\chi_{\alpha\beta} = 3\boldsymbol{k}_\beta\cdot(\boldsymbol{v}_0 - \boldsymbol{v})/h, \qquad 2\omega_{\alpha\beta} = 5\boldsymbol{k}_\beta\cdot(\Delta\boldsymbol{v} - \boldsymbol{\psi})/h, \qquad 2\varepsilon_{\alpha\gamma} = \frac{\boldsymbol{k}_\alpha}{A_\gamma}\cdot\frac{\partial \boldsymbol{v}}{\partial\gamma} + \frac{1}{A_\gamma}\frac{\partial W}{\partial\gamma} + \frac{1}{h}\boldsymbol{k}_\gamma\cdot\Delta\boldsymbol{v},$$
$$2\chi_{\alpha\gamma} = 3\boldsymbol{k}_\gamma\cdot(\boldsymbol{v}_0 - \boldsymbol{v})/h, \qquad 2\omega_{\alpha\gamma} = 5\boldsymbol{k}_\gamma\cdot(\Delta\boldsymbol{v} - \boldsymbol{\psi})/h.$$

Let us express the forces $\boldsymbol{N}_\beta$ and $\boldsymbol{N}_\gamma$ and moments $\boldsymbol{M}_\beta$ and $\boldsymbol{M}_\gamma$ in terms of strains and curvatures. We substitute the strain approximations (3.4) and (3.5) into (2.2) and the expressions for stresses into (3.2). Integrating the resultant relations with allowance for the orthogonal property of Legendre polynomials, after some manipulations, we obtain

$$\boldsymbol{N}_\beta = 2h\Big[\frac{5}{6}\Big(2\mu\varepsilon'_{\alpha\beta} + \frac{1}{5}(\boldsymbol{P}_0\cdot\boldsymbol{k}_\beta)\Big)\boldsymbol{k}_\alpha + \Big(\frac{E}{1-\nu^2}(\varepsilon_{\beta\beta} + \nu\varepsilon_{\gamma\gamma}) + \frac{\nu}{1-\nu}(\boldsymbol{P}_0\cdot\boldsymbol{k}_\alpha)\Big)\boldsymbol{k}_\beta + 2\mu\Big(\varepsilon_{\beta\gamma} + \frac{h}{3R_\gamma}\chi_{\beta\gamma}\Big)\boldsymbol{k}_\gamma\Big],$$
$$\boldsymbol{N}_\gamma = 2h\Big[\frac{5}{6}\Big(2\mu\varepsilon'_{\alpha\gamma} + \frac{1}{5}(\boldsymbol{P}_0\cdot\boldsymbol{k}_\gamma)\Big)\boldsymbol{k}_\alpha + \Big(\frac{E}{1-\nu^2}(\varepsilon_{\gamma\gamma} + \nu\varepsilon_{\beta\beta}) + \frac{\nu}{1-\nu}(\boldsymbol{P}_0\cdot\boldsymbol{k}_\alpha)\Big)\boldsymbol{k}_\gamma + 2\mu\Big(\varepsilon_{\beta\gamma} + \frac{h}{3R_\beta}\chi_{\beta\gamma}\Big)\boldsymbol{k}_\beta\Big],$$
$$\boldsymbol{M}_\beta = \frac{2h^2}{3}\Big[\Big(\frac{E}{1-\nu^2}(\chi_{\beta\beta} + \nu\chi_{\gamma\gamma}) + \frac{\nu}{1-\nu}(\Delta\boldsymbol{P}\cdot\boldsymbol{k}_\alpha)\Big)\boldsymbol{k}_\beta + 2\mu\chi_{\beta\gamma}\boldsymbol{k}_\gamma\Big], \tag{3.6}$$
$$\boldsymbol{M}_\gamma = \frac{2h^2}{3}\Big[\Big(\frac{E}{1-\nu^2}(\chi_{\beta\beta} + \nu\chi_{\gamma\gamma}) + \frac{\nu}{1-\nu}(\Delta\boldsymbol{P}\cdot\boldsymbol{k}_\alpha)\Big)\boldsymbol{k}_\gamma + 2\mu\chi_{\beta\gamma}\boldsymbol{k}_\beta\Big].$$

Here E is Young's modulus and ν is Poisson's ratio,

$$2\varepsilon'_{\alpha\beta} = \frac{\boldsymbol{k}_\alpha}{A_\beta}\cdot\frac{\partial \boldsymbol{v}}{\partial\beta} + \frac{1}{A_\beta}\frac{\partial W}{\partial\beta} + \frac{\boldsymbol{k}_\beta\cdot\boldsymbol{\psi}}{h}; \qquad 2\varepsilon'_{\alpha\gamma} = \frac{\boldsymbol{k}_\alpha}{A_\gamma}\cdot\frac{\partial \boldsymbol{v}}{\partial\gamma} + \frac{1}{A_\gamma}\frac{\partial W}{\partial\gamma} + \frac{\boldsymbol{k}_\gamma\cdot\boldsymbol{\psi}}{h}.$$

We find the relation between the external surface forces $\boldsymbol{P}^\pm$ and displacements. We insert the strain approximations (3.4) and (3.5) into (2.2) and the resultant expressions for stresses into (3.2). Using the properties of Legendre polynomials for $\xi = \pm 1$, we obtain

$$\Delta\boldsymbol{q} = 3\mu(\boldsymbol{v}_0 - \boldsymbol{v})/h, \qquad \boldsymbol{q}_0 = 5\mu(\Delta\boldsymbol{v} - \boldsymbol{\psi})/h,$$
$$g_0 = E\Delta W/h + \nu N/(2h), \qquad \Delta g = 3E(W_0 - W)/h + 3\nu M/(2h^2). \tag{3.7}$$

Here

$$\Delta\boldsymbol{q} = (\boldsymbol{q}^+ - \boldsymbol{q}^-)/2, \qquad \boldsymbol{q}_0 = (\boldsymbol{q}^+ + \boldsymbol{q}^-)/2, \qquad \Delta g = (g^+ - g^-)/2, \qquad g_0 = (g^+ + g^-)/2,$$
$$N = \boldsymbol{N}_\beta\cdot\boldsymbol{k}_\beta + \boldsymbol{N}_\gamma\cdot\boldsymbol{k}_\gamma, \qquad M = \boldsymbol{M}_\beta\cdot\boldsymbol{k}_\beta + \boldsymbol{M}_\gamma\cdot\boldsymbol{k}_\gamma, \qquad \boldsymbol{q}^\pm = (\sigma_{\alpha\beta}\boldsymbol{k}_\beta + \sigma_{\alpha\gamma}\boldsymbol{k}_\gamma)\Big|_{\xi=\pm1}, \qquad g^\pm = \sigma_{\alpha\alpha}\Big|_{\xi=\pm1}.$$

The equations of equilibrium have the form

$$\frac{\partial}{\partial\beta}(A_\gamma\boldsymbol{N}_\beta) + \frac{\partial}{\partial\gamma}(A_\beta\boldsymbol{N}_\gamma) + 2A_\beta A_\gamma\Delta\boldsymbol{P} = 0,$$
$$\frac{\partial}{\partial\beta}(A_\gamma\boldsymbol{k}_\alpha\times\boldsymbol{M}_\beta) + \frac{\partial}{\partial\gamma}(A_\beta\boldsymbol{k}_\alpha\times\boldsymbol{M}_\gamma) + A_\beta A_\gamma(\boldsymbol{k}_\beta\times\boldsymbol{N}_\beta + \boldsymbol{k}_\gamma\times\boldsymbol{N}_\gamma) + A_\beta A_\gamma 2h(\boldsymbol{k}_\alpha\times\boldsymbol{P}_0) = 0. \tag{3.8}$$

4. Equations of a Laminated Body Composed of Parallel Layers. We consider the surface S_0 formed by revolution of a convex curve L_0 with a support function $F_0(\gamma)$. Curves L_i $(i = \overline{1,n})$ with support functions $F_i(\gamma) = F_{i-1}(\gamma) + 2h_i$ form a family of equidistant surfaces S_i, the distance between the neighboring surfaces S_i, S_{i-1} being equal to $2h_i$.

Let B be a laminated body composed of monolayers B_i $(i = \overline{1,n})$ bounded by the surfaces S_{i-1}, S_i. We denote the quantities that refer to the layer B_i by the superscript i. From the algebraic equations (3.7), we obtain the expressions for $(\boldsymbol{U}^+)^i$ and $(\boldsymbol{P}^+)^i$:

$$\begin{gathered}(g^+)^i = 3E^i(W^i - (W^-)^i)/h^i - 2(g^-)^i + 3\nu^i(N^i - M^i/h^i)/(2h^i),\\ (\boldsymbol{q}^+)^i = 15\mu^i((\boldsymbol{v}^-)^i + \boldsymbol{\psi}^i - \boldsymbol{v}^i)/h^i + 4(\boldsymbol{q}^-)^i - 3\boldsymbol{Q}^i/(2h^i),\\ (W^+)^i = -2(W^-)^i + 3W^i - h^i(g^-)^i/E^i + \nu^i(N^i - 3M^i/h^i)/(2E^i),\\ (\boldsymbol{v}^+)^i = 4(\boldsymbol{v}^-)^i + 5\boldsymbol{\psi}^i - 3\boldsymbol{v}^i + h^i(\boldsymbol{q}^-)^i/\mu^i - \boldsymbol{Q}^i/(2\mu^i).\end{gathered} \tag{4.1}$$

The following continuity conditions for stresses and displacements should hold at the interlayer-contact surfaces S_i $(i = \overline{1,n-1})$:

$$(\boldsymbol{q}^+)^i = (\boldsymbol{q}^-)^{i+1}, \qquad (g^+)^i = (g^-)^{i+1} \tag{4.2}$$

and

$$(\boldsymbol{v}^+)^i = (\boldsymbol{v}^-)^{i+1}, \qquad (W^+)^i = (W^-)^{i+1}. \tag{4.3}$$

Below, we confine our attention to the case where the following stresses are specified at the surfaces S_0 and S_n of the layered body B:

$$(\boldsymbol{q}^-)^1 = \boldsymbol{Q}_0, \quad (\boldsymbol{q}^+)^n = \boldsymbol{Q}_n, \quad (g^-)^1 = G_0, \quad (g^+)^n = G_n. \tag{4.4}$$

Equations (4.1)–(4.4) are a system of linear algebraic equations for displacements and stresses at the interlayer-contact surfaces S_i $(i = \overline{1,n-1})$ and displacements at the front faces S_0 and S_n. Solving this system, we obtain

$$\begin{gathered}(g^+)^i = A_1^i G_n + A_2^i G_0 + \sum_{k=1}^{i}(a_{1k}^i W^k + a_{2k}^i N^k + a_{3k}^i M^k),\\ (W^+)^i = B_1^i G_n + B_2^i G_0 + \sum_{k=1}^{i}(b_{1k}^i W^k + b_{2k}^i N^k + b_{3k}^i M^k),\\ (\boldsymbol{q}^+)^i = C_1^i \boldsymbol{Q}_n + C_2^i \boldsymbol{Q}_0 + \sum_{k=1}^{i}(c_{1k}^i \boldsymbol{v}^k + c_{2k}^i \boldsymbol{\psi}^k + c_{3k}^i \boldsymbol{Q}^k),\\ (\boldsymbol{v}^+)^i = D_1^i \boldsymbol{Q}_n + D_2^i \boldsymbol{Q}_0 + \sum_{k=1}^{i}(d_{1k}^i \boldsymbol{v}^k + d_{2k}^i \boldsymbol{\psi}^k + d_{3k}^i \boldsymbol{Q}^k),\\ W_n = (W^+)^n = B_1^n G_n + B_2^n G_0 + \sum_{k=1}^{n}(b_{1k}^n W^k + b_{2k}^n N^k + b_{3k}^n M^k),\\ W_0 = (W^-)^1 = B_1^0 G_n + B_2^0 G_0 + \sum_{k=1}^{n}(b_{1k}^0 W^k + b_{2k}^0 N^k + b_{3k}^0 M^k),\\ \boldsymbol{V}_n = (\boldsymbol{v}^+)^n = D_1^n \boldsymbol{Q}_n + D_2^n \boldsymbol{Q}_0 + \sum_{k=1}^{n}(d_{1k}^n \boldsymbol{v}^k + d_{2k}^n \boldsymbol{\psi}^k + d_{3k}^n \boldsymbol{Q}^k),\\ \boldsymbol{V}_0 = (\boldsymbol{v}^-)^1 = D_1^0 \boldsymbol{Q}_n + D_2^0 \boldsymbol{Q}_0 + \sum_{k=1}^{n}(d_{1k}^0 \boldsymbol{v}^k + d_{2k}^0 \boldsymbol{\psi}^k + d_{3k}^0 \boldsymbol{Q}^k).\end{gathered} \tag{4.5}$$

Substituting expressions (4.5) into formulas (3.5)–(3.8), after some transformations, we obtain the following system of partial differential equations:

$$\frac{\partial}{\partial\beta}(G_1\boldsymbol{X}) + \frac{\partial}{\partial\gamma}(G_2\boldsymbol{X}) = G_3\boldsymbol{X} + G_4. \tag{4.6}$$

Here G_k $(k = 1, \ldots, 4)$ are $10n \times 10n$ matrices and $\boldsymbol{X} = (\boldsymbol{v}^i, \boldsymbol{\psi}^i, W^i, \boldsymbol{N}^i, \boldsymbol{M}^i)$ $(i = 1, \ldots, n)$ is the vector of unknown functions.

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